Potential Theory and Nonlinear Elliptic Equations
Lecture 2

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Publications

1. A. Grigor'yan and I. Verbitsky, *Pointwise estimates of solutions to nonlinear equations for non-local operators*, ***Ann. Scuola Norm. Super. Pisa***, **20** (2020) 721–750
2. A. Grigor'yan and I. Verbitsky, *Pointwise estimates of solutions to semilinear elliptic equations and inequalities*, ***J. D′Analyse Math.***, **137** (2019) 529–558
3. A. Grigor'yan and W. Hansen, *Lower estimates for a perturbed Green function*, ***J. D′Analyse Math.***, **104** (2008), 25–58.
4. N. Kalton and I. Verbitsky, *Nonlinear equations and weighted norm inequalities*, ***Trans. Amer. Math. Soc.*** **351** (1999), 3441–3497.
5. H. Brezis and X. Cabré, *Some simple nonlinear PDE's without solutions*, ***Boll. Unione Mat. Ital.***, **8**, Ser. 1-B (1998) 223–262.

Additional literature

1. J. L. Doob, *Classical Potential Theory and Its Probabilistic Counterpart,* Classics in Math., Springer, New York –Berlin–Heidelberg–Tokyo, 2001 (Reprint of the 1984 ed.)

2. A. Grigor'yan, *Heat Kernel and Analysis on Manifolds*, Amer. Math.Soc./Intern. Press Studies in Adv. Math., **47**, 2009.

3. N. S. Landkof, *Foundations of Modern Potential Theory*, Grundlehren der math. Wissenschaften, **180**, Springer, New York–Heidelberg, 1972.

The Laplace-Beltrami operator

Recall that the gradient operator ∇ is defined by

$$(\nabla u)^i = \sum_{j=1}^{n} g^{ij} \partial_{x_j} u.$$

The divergence operator $\mathbf{div}$ on vector fields F^i is defined by

$$\mathrm{div} F = \frac{1}{\sqrt{\det g}} \sum_{i=1}^{n} \partial_{x_i} \left(\sqrt{\det g}\, F^i \right).$$

The Laplace-Beltrami operator $\mathcal{L}_0$ is represented in the form

$$\mathcal{L}_0 = \mathrm{div} \circ \nabla.$$

The weighted Laplace operator

Let (M, m) be a weighted manifold with $dm = \omega \, dm_0$.

The weighted divergence operator is defined by

$$\mathrm{div}_\omega = \frac{1}{\omega} \circ \mathrm{div} \circ \omega.$$

Recall that ∇ and div are the Riemannian gradient and divergence, respectively, and do not depend on the weight ω.

The (weighted) Laplace operator $\mathcal{L} = \Delta$ is defined by $\Delta = \mathrm{div}_\omega \circ \nabla$.
From the definitions of ∇ and div, it follows that

$$\Delta u = \frac{1}{\omega} \mathrm{div}\,(\omega \nabla u) = \frac{1}{\omega \sqrt{\det g}} \sum_{i,j=1}^{n} \partial_{x_i} \left(\omega \sqrt{\det g}\, g^{ij} \partial_{x_j} u \right), \quad (1)$$

acting on C^2 functions u on M.

Example (elliptic differential operators in $\mathbb{R}^n$)

In an open set $\Omega \subseteq \mathbb{R}^n$ consider the operator

$$Lu = b(x) \sum_{i,j=1}^{n} \partial_{x_i} \left(a_{ij}(x) \, \partial_{x_j} u \right), \tag{2}$$

where b, $A = (a_{ij})$ are smooth functions, and $b > 0$.

We assume here that the matrix $A(x)$ is symmetric and positive definite for any $x \in \Omega$.

In other words, the operator L is elliptic (the uniform ellipticity is not needed).

Example (elliptic differential operators in $\mathbb{R}^n$)

(continuation)

We claim that L coincides with the *weighted* Laplace operator Δ on $\Omega \subseteq \mathbb{R}^n$ with the Riemannian metric g and weight ω chosen so that

$$\left(g^{ij}\right) = b\left(a_{ij}\right), \qquad \omega = b^{\frac{n}{2}-1}\sqrt{\det A}. \tag{3}$$

Clearly,

$$\det g = \det\left(g_{ij}\right) = \frac{1}{b^n \det A}. \tag{4}$$

The measure $dm = \omega\, dm_0$ associated with Δ is given by

$$dm = \omega\sqrt{\det g}\, dx = b^{\frac{n}{2}-1}\sqrt{\det A}\frac{1}{\sqrt{b^n \det A}}\, dx = \frac{1}{b}\, dx, \tag{5}$$

where dx is Lebesgue measure.

Example (elliptic differential operators in $\mathbb{R}^n$)

(continuation)

Recall that by (1), we have

$$\Delta u = \frac{1}{\omega \sqrt{\det g}} \sum_{i,j=1}^{n} \partial_{x_i} \left(\omega \sqrt{\det g}\, g^{ij} \partial_{x_j} u \right). \qquad (6)$$

Substituting (3), (4) into (6) yields

$$\begin{aligned} \Delta u &= \frac{\sqrt{b^n \det A}}{b^{\frac{n}{2}-1}\sqrt{\det A}} \sum_{i,j=1}^{n} \partial_{x_i} \left(b^{\frac{n}{2}-1}\sqrt{\det A} \frac{1}{\sqrt{b^n \det A}} b a^{ij} \partial_{x_j} u \right) \\ &= b \sum_{i,j=1}^{n} \partial_{x_i} \left(a_{ij}(x)\, \partial_{x_j} u \right) = Lu. \end{aligned}$$

Therefore, the results below for a general weighted manifold (M, m), are applicable to the operator L in $\Omega \subset \mathbb{R}^n$ with the measure m. In particular, if $b \equiv 1$, then $L = \mathrm{div}(A \nabla \cdot)$ and m is Lebesgue measure.

The Doob transform

Given a positive C^2 function h in $\Omega \subseteq M$, consider the following operator,

$$L^h = \frac{1}{h} \circ \Delta \circ h$$

acting on $C^2(\Omega)$. The operator L^h is called the Doob transform of Δ. Usually it is used for harmonic functions h, but we use L^h for **superharmonic** h as well [Grigor'yan-Verbitsky 2019].
Notice that L^h can be written in the form

$$L^h v = \Delta^h v + \frac{\Delta h}{h} v, \tag{7}$$

where $v \in C^2(\Omega)$ and Δ^h is the h-Laplacian defined by

$$\Delta^h v = \frac{1}{h^2} \mathrm{div}_\omega (h^2 \nabla v). \tag{8}$$

Note that Δ^h is the Laplace operator for the measure $h^2 dm = h^2 \omega \, dm_0$.

Green functions

Recall that, for a general weight ω, the Laplace operator $\mathcal{L} = \Delta$ is *symmetric* with respect to the measure m. Moreover, Δ satisfies the Chain Rule and the Product Rule, like in the case $\omega = 1$, when $\Delta = \mathcal{L}_0$ is the Laplace-Beltrami operator.

For any open connected set $\Omega \subseteq M$, we denote by $G^{\Omega}(x, y)$ the infimum of all positive fundamental solutions of Δ in Ω.

Then the following is true:

either $G^{\Omega}(x, y) \equiv +\infty$ or $G^{\Omega}(x, y) < +\infty$ for all $x \neq y$.

In the latter case we will say that G^{Ω} is ***non-trivial***, and call G^{Ω} the ***minimal Green function*** (positive, symmetric) of Δ in Ω.

The existence of a non-trivial G^{Ω} is the only assumption on Ω that we impose.

Green potentials

If G^Ω is the non-trivial minimal Green function, then for any $\mu \in \mathcal{M}^+(\Omega)$, the Green potential $\mathrm{G}^\Omega\mu$ is defined by

$$\mathrm{G}^\Omega\mu(x) = \int_\Omega G^\Omega(x, y)\, d\mu(y)\,.$$

For a nonnegative $f \in L^1_{\mathrm{loc}}(\Omega, m)$, we set $\mathrm{G}^\Omega f := \mathrm{G}^\Omega(f\, dm)$.
For a signed function $f \in L^1_{\mathrm{loc}}(\Omega, m)$,

$$\mathrm{G}^\Omega f(x) = \mathrm{G}^\Omega f_+(x) - \mathrm{G}^\Omega f_-(x)$$

assuming at least one of the following:

$$\mathrm{G}^\Omega f_+(x) < +\infty, \quad \text{or} \quad \mathrm{G}^\Omega f_-(x) < +\infty.$$

Then $\mathrm{G}^\Omega f(x)$ is said to be ***well-defined***.

Remark. If Ω is relatively compact then G^Ω is non-trivial, $G^\Omega(x, \cdot) \in L^1(\Omega)$, and $\mathrm{G}^\Omega f$ is finite for any $f \in L^\infty(\Omega)$.

Local case: semi-linear inequalities

(with boundary conditions)

Our main goal is to obtain "sharp" pointwise estimates of positive sub/super-solutions to the following model semi-linear problem.

Problem. Let $\Omega \subset M$ be an open relatively compact connected subdomain of M. Given $V \in C(\overline{\Omega})$, $\mu \in C\left(\overline{\Omega}\right)$, $\nu \in C(\partial\Omega)$, $\mu, \nu \geq 0$, ***assume*** that there exists a ***nonnegative*** solution u to the following semi-linear Dirichlet problem:

$$\begin{cases} -\Delta u + V\, u^q \geq \mu & \text{in } \Omega \\ u \geq \nu & \text{in } \partial\Omega, \end{cases} \tag{9}$$

if $q > 0$, and

$$\begin{cases} -\Delta u + V\, u^q \leq \mu & \text{in } \Omega \\ u \leq \nu & \text{in } \partial\Omega, \end{cases} \tag{10}$$

if $q < 0$.

Remark. Here $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ is a ***classical*** solution.

The auxiliary linear Dirichlet problem

Remark. Analogues for **general** domains $\Omega \subseteq M$ (not necessarily relatively compact) and **non-smooth** coefficients/data are discussed below.

We will compare u to the solution h of the following ***auxiliary*** linear Dirichlet problem:

$$\begin{cases} -\Delta h = \mu & \text{in } \Omega, \\ h = \nu & \text{in } \partial\Omega, \end{cases}$$

where $h \geq 0$ is ***superharmonic*** in Ω $(\mu, \nu \geq 0)$, for regular domains Ω. We will write

$$h = \mathrm{P}^{\Omega}\nu + \mathrm{G}^{\Omega}\mu.$$

For smooth domains $\mathrm{P}^{\Omega}\nu$ and $\mathrm{G}^{\Omega}\mu$ are given by the Poisson and Green integrals respectively.

Main results: local case

Theorem 3 (Grigor'yan-Verbitsky 2019)

Let (M, m) be a weighted manifold, $\Omega \subset M$ an open relatively compact subdomain of M, $\partial\Omega$ regular, $V \in C(\overline{\Omega})$, $\mu \in C\left(\overline{\Omega}\right)$, $\nu \in C(\partial\Omega)$, $\mu, \nu \geq 0$, μ locally Hölder continuous, either $\mu \not\equiv 0$ or $\nu \not\equiv 0$, which ensures that $h > 0$ in Ω.
Suppose $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ is a non-negative super-solution to (9) if $q > 0$, or sub-solution to (10) if $q < 0$.
Then the following statements hold for all $x \in \Omega$.

(i) If $q = 1$, then

$$u(x) \geq h(x) e^{-\frac{1}{h(x)} \mathrm{G}^{\Omega}(hV)(x)}. \tag{11}$$

Main results: local case (continuation)

Theorem 3 (statements (ii), (iii))

(ii) If $\boldsymbol{q > 1}$*, then* **necessarily** *the condition*

$$-(q-1)\, G^{\Omega}(h^q V)(x) < h(x) \tag{12}$$

holds in $\boldsymbol{\Omega}$*, and*

$$u(x) \geq h(x) \left[1 + (q-1) \frac{G^{\Omega}(h^q V)(x)}{h(x)} \right]^{-\frac{1}{q-1}}. \tag{13}$$

(iii) If $\boldsymbol{0 < q < 1}$*, then*

$$u(x) \geq h(x) \left[1 - (1-q) \frac{G^{\Omega}(\chi_{\Omega^+} h^q V)(x)}{h(x)} \right]_+^{\frac{1}{1-q}}, \tag{14}$$

where $\boldsymbol{\Omega^+ = \{x \in \Omega : \ u(x) > 0\}}$.

Main results: local case

(continuation)

Theorem 3 (statement (iv))

(iv) If $q < 0$ and $u > 0$ in Ω then **necessarily** *the condition*

$$(1-q)\,\mathrm{G}^{\Omega}(h^q V)(x) < h(x), \tag{15}$$

holds in Ω, and

$$u(x) \le h(x)\left[1-(1-q)\frac{\mathrm{G}^{\Omega}(h^q V)(x)}{h(x)}\right]^{\frac{1}{1-q}}, \tag{16}$$

provided $\mathrm{G}^{\Omega}(h^q V)(x)$ is well-defined.

Inequalities for $L^h v$; $v = \phi^{-1}\left(\frac{u}{h}\right)$, ϕ increasing

Lemma (inequalities for the Doob transform)

Let h be a positive C^2-function in Ω. Let u be a solution of

$$-\Delta u + V\, u^q \geq -\Delta h \qquad (17)$$

in Ω, where $V \in C(\Omega)$ and $q \in \mathbb{R} \setminus \{0\}$. Let ϕ be a C^2 function on an interval $I \subset \mathbb{R}$ such that $\phi' > 0$ in I. Assume $\frac{u}{h}(\Omega) \subset \phi(I)$.
Then $v = \phi^{-1}\left(\frac{u}{h}\right)$ satisfies the differential inequality:

$$-L^h v + h^{q-1} V \frac{\phi(v)^q}{\phi'(v)} \geq L^h 1 \left(\frac{\phi(v) - 1}{\phi'(v)} - v \right) + \frac{\phi''(v)}{\phi'(v)} |\nabla v|^2. \qquad (18)$$

If in place of (17) we have

$$-\Delta u + V u^q \leq -\Delta h, \qquad (19)$$

then (18) holds with $\leq$ instead of $\geq$.

Proof of the lemma

Recall that $L^h = \frac{1}{h} \circ \Delta \circ h$. In particular, $L^h 1 = \frac{\Delta h}{h}$.
Set $\tilde{u} = \frac{u}{h}$, so that $L^h \tilde{u} = \frac{1}{h} \Delta u$. Divide both sides of (17) by h:

$$-L^h \tilde{u} + h^{q-1} V \tilde{u}^q \geq -L^h 1. \qquad (20)$$

By the Chain Rule, for any $v \in C^2(\Omega)$

$$\Delta^h \phi(v) = \phi'(v) \Delta^h v + \phi''(v) |\nabla v|^2.$$

By (7) applied to $\tilde{u} = \phi(v)$, we have $L^h \tilde{u} = \Delta^h \tilde{u} + \frac{\Delta h}{h} \tilde{u}$. Hence,

$$\begin{aligned} L^h \phi(v) &= \Delta^h \phi(v) + \frac{\Delta h}{h} \phi(v) \\ &= \phi'(v) \Delta^h v + \phi''(v) |\nabla v|^2 + \frac{\Delta h}{h} \phi(v) \\ &= \phi'(v) (\Delta^h v + \frac{\Delta h}{h} v) + \phi''(v) |\nabla v|^2 + \frac{\Delta h}{h} \left(\phi(v) - v \phi'(v) \right) \\ &= \phi'(v) L^h v + \phi''(v) |\nabla v|^2 + \frac{\Delta h}{h} \left(\phi(v) - v \phi'(v) \right). \end{aligned}$$

End of the proof

Therefore, solving for $L^h v$, we have

$$- L^h v = -\frac{L^h \phi(v)}{\phi'(v)} + \frac{\phi''(v)}{\phi'(v)}|\nabla v|^2 + \frac{\Delta h}{h}\left(\frac{\phi(v)}{\phi'(v)} - v\right). \qquad (21)$$

Since $\tilde{u} = \phi(v)$, it follows that (20) yields the following estimate:

$$-L^h \phi(v) + h^{q-1} V \phi(v)^q \geq -L^h 1.$$

Substituting this inequality into (21), we get rid of $L^h \phi(v)$:

$$-L^h v + h^{q-1} V \frac{\phi(v)^q}{\phi'(v)} \geq L^h 1 \left(\frac{\phi(v) - 1}{\phi'(v)} - v\right) + \frac{\phi''(v)}{\phi'(v)}|\nabla v|^2.$$

This proves the desired inequality for $L^h v$.

The converse inequality with $\leq$ in place $\geq$ is proved in the same way. □

Inequalities for $\Delta(hv)$; $v = \phi^{-1}\left(\frac{u}{h}\right)$

ϕ increasing, convex

Corollary (superharmonic h)

Under the hypotheses of the Lemma, assume in addition $\Delta h \le 0$ in Ω and $0 \in I$.
(i) If ϕ is convex in the interval I, so that

$$\phi(0) = 1, \quad \phi' > 0, \quad \phi'' \ge 0, \tag{22}$$

and u satisfies $-\Delta u + V u^q \ge -\Delta h$, then the function $v = \phi^{-1}\left(\frac{u}{h}\right)$ satisfies the following inequality in Ω:

$$-\Delta(hv) + h^q V \frac{\phi(v)^q}{\phi'(v)} \ge 0. \tag{23}$$

Inequalities for $\Delta(hv)$; $v = \phi^{-1}\left(\frac{u}{h}\right)$

ϕ increasing, concave

Corollary (superharmonic h)

(ii) If ϕ is concave in the interval I, so that

$$\phi(0) = 1, \quad \phi' > 0, \quad \phi'' \leq 0, \tag{24}$$

and u satisfies $-\Delta u + V u^q \leq -\Delta h$, then v satisfies

$$-\Delta(hv) + h^q V \frac{\phi(v)^q}{\phi'(v)} \leq 0. \tag{25}$$

Proof of the corollary

To prove (i), notice that, for a convex ϕ such that $\phi' > 0$, $\phi(0) = 1$,

$$\frac{\phi(v) - 1}{\phi'(v)} - v \geq 0,$$

since the chord of the graph of the convex function ϕ between the points $(0, 1)$ and $(v, \phi(v))$ lies above the tangent line at $(v, \phi(v))$.
Using also that $L^h 1 = \frac{\Delta h}{h} \leq 0$, we obtain from the Lemma:

$$-L^h v + h^{q-1} V \frac{\phi(v)^q}{\phi'(v)} \geq 0,$$

which is equivalent to (23), since $\Delta(hv) = h\, L^h v$.
The proof of statement (ii) is similar. □

A comparison principle for superharmonic functions

The following two lemmas enable us to get rid of some technical assumptions like $\inf_\Omega h > 0$ initially used in the proofs below.

Lemma (a comparison principle)

Suppose $\Omega \subseteq M$ is open, and F is a superharmonic function in Ω. Suppose $F = F_1 + F_2$ where $\liminf_{x\to z} F_1(x) \ge 0$ for every $z \in \partial_\infty \Omega$, and $F_2 \ge -P$, where $P = G^\Omega \mu$ is a Green potential of a positive measure μ in Ω, $P \not\equiv +\infty$ on every component of Ω. Then $F \ge 0$ in Ω.

Proof.

The function $F + P$ is obviously superharmonic, and $F + P \ge F_1$. Hence $\liminf_{x\to z}(F + P)(x) \ge 0$ for $z \in \partial_\infty \Omega$, and by the maximum principle $F + P \ge 0$ on Ω. Hence F is a superharmonic majorant of $-P$, whose least superharmonic majorant must be zero, which yields $F \ge 0$. □

Remark. The case $P = 0$ gives the usual form of the maximum principle.

A version of the maximum principle

The following version of the maximum principle will be frequently used below. It is deduced from the previous comparison lemma.

Lemma (a maximum principle)

Let $\boldsymbol{\Omega}$ *be an open subset of* $\boldsymbol{M}$ *with non-trivial Greeen's function, and let* $\boldsymbol{v} \in \boldsymbol{C}^2(\boldsymbol{\Omega})$ *satisfy*

$$\begin{cases} -\boldsymbol{\Delta v} \geq \boldsymbol{f} & \textit{in } \boldsymbol{\Omega}, \\ \liminf_{x \to \partial_\infty \Omega} \boldsymbol{v}(\boldsymbol{x}) \geq \boldsymbol{0}, \end{cases}$$

where $\boldsymbol{f} \in \boldsymbol{C}(\boldsymbol{\Omega})$ *such that* $\mathrm{G}^\Omega \boldsymbol{f}$ *is well defined in* $\boldsymbol{\Omega}$*. Then*

$$\boldsymbol{v}(\boldsymbol{x}) \geq \mathrm{G}^\Omega \boldsymbol{f}(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in \boldsymbol{\Omega}. \tag{26}$$

Semi-linear problems in "nice" domains

under the assumption $\inf_\Omega h > 0$

Lemma (proof of Theorem 3: $\inf_\Omega h > 0$, smooth boundary)

Suppose Ω is a relatively compact domain in M with smooth boundary. Suppose $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$, $V \in C\left(\overline{\Omega}\right)$, and μ, ν are non-negative functions such that $\nu \in C(\partial\Omega)$, and $\mu \in C\left(\overline{\Omega}\right) \cap C^\alpha(\Omega)$ for some $\alpha \in (0, 1]$. Let

$$h = \mathrm{P}^\Omega \nu + \mathrm{G}^\Omega \mu. \tag{27}$$

If $\inf_\Omega h > 0$, then the following statements hold.
(i) In the case $q > 0$, if $u > 0$ is a solution of

$$\begin{cases} -\Delta u + V u^q \geq \mu & \text{in } \Omega, \\ u \geq \nu & \text{in } \partial\Omega, \end{cases} \tag{28}$$

then statements (i)-(iii) of Theorem 3 are valid (lower bounds for u).

Semi-linear problems in "nice" domains

under the assumption $\inf_\Omega h > 0$

Lemma (continuation)

(ii) In the case $q < 0$, if $u > 0$ is a solution of

$$\begin{cases} -\Delta u + V u^q \leq \mu & \text{in } \Omega, \\ u \leq \nu & \text{in } \partial\Omega, \end{cases} \tag{29}$$

then statement (iv) of Theorem 3 is valid (upper bounds for u).

Remarks. 1. The technical assumption $\inf_\Omega h > 0$ is removed using the maximum principle lemma stated above.

2. The restriction that Ω has a smooth boundary is unnecessary, and will be removed below.

Proof of the Lemma

By the hypotheses, $h \in C^2(\Omega)$, $-\Delta h = \mu$, and $h > 0$ in Ω. Choose the function ϕ in the Corollary to satisfy the equation

$$\phi'(v) = \phi(v)^q. \tag{30}$$

For $q = 1$, this gives

$$\phi(v) = e^v, \quad v \in \mathbb{R}, \tag{31}$$

while for $q \neq 1$, we obtain

$$\phi(v) = [(1-q)v + 1]^{\frac{1}{1-q}}, \quad v \in I_q, \tag{32}$$

where the domain I_q of ϕ is given by:

$$I_q = \begin{cases} (-\frac{1}{1-q}, +\infty) & \text{if } q < 1, \\ (-\infty, +\infty) & \text{if } q = 1, \\ (-\infty, \frac{1}{q-1}) & \text{if } q > 1. \end{cases} \tag{33}$$

Proof of the Lemma

(continuation)

Note that in all cases $\phi\left(I_q\right) = (0, \infty)$. Also, we have

$$\phi'(v) = [(1-q)v+1]^{\frac{q}{1-q}}, \quad \phi''(v) = q[(1-q)v+1]^{\frac{2q-1}{1-q}}. \tag{34}$$

Since $u = h\phi\left(v\right)$, all the estimates in the case $q > 0$ follow from:

$$v(x) \geq -\frac{1}{h(x)} \mathrm{G}^{\Omega}(h^q V)(x) \text{ for all } x \in \Omega. \tag{35}$$

For $q < 0$, we will have the opposite inequality.
Let us use the function hv expressed explicitly via u and h as follows:

$$hv = \begin{cases} \frac{1}{q-1} h\left(1 - (\frac{h}{u})^{q-1}\right) & \text{if } 1 < q < +\infty, \\ h \log(\frac{u}{h}) & \text{if } q = 1, \\ \frac{1}{1-q}\left(h^q u^{1-q} - h\right) & \text{if } -\infty < q < 1. \end{cases} \tag{36}$$

Proof of the Lemma

(continuation)

Since $u > 0$, $h > 0$ in Ω, we have $\frac{u}{h}(\Omega) \subset \phi(I_q) = (0, \infty)$, and $hv \in C^2(\Omega)$.

In the case $q > 0$ the function ϕ is concave, increasing, and $\phi(0) = 1$.
We obtain from the Corollary,

$$-\Delta (hv) + h^q V \geq 0. \tag{37}$$

Since $u \geq \nu > 0$ on $\partial\Omega$, and consequently $\inf_\Omega u > 0$, we actually have $hv \in C\left(\overline{\Omega}\right) \cap C^2(\Omega)$, and $hv \geq 0$ on $\partial\Omega$, which by the maximum principle implies (35).

In addition, if $q > 1$, then $I_q = (-\infty, \frac{1}{q-1})$, so that $v(x) < \frac{1}{q-1}$.
Combining this estimate with (35) gives the **necessary** condition for the existence of u:

$$-\mathrm{G}^\Omega(h^q V)(x) < \frac{1}{q-1} h(x), \quad \text{for all } x \in \Omega.$$

Proof of the Lemma

(continuation)

Similarly, in the case $q < 0$, we have $hv \in C\left(\overline{\Omega}\right) \cap C^2(\Omega)$ since $\inf_\Omega h > 0$. The inequality $u \leq \nu$ on $\partial\Omega$ yields the boundary condition $hv \leq 0$ on $\partial\Omega$. By the Corollary we obtain that in Ω,

$$-\Delta(hv) + h^q V \leq 0, \quad \text{for all } x \in \Omega. \tag{38}$$

Together with the boundary condition this yields by the maximum principle

$$v(x) \leq -\frac{1}{h(x)} \mathrm{G}^\Omega(h^q V)(x), \quad \text{for all } x \in \Omega. \tag{39}$$

In view of (36), this translates into the desired inequality (16) for u. Moreover, since $I_q = \left(-\frac{1}{1-q}, +\infty\right)$, in this case $v(x) > -\frac{1}{1-q}$. Combining this estimate with (39) yields the **necessary** condition (15) for the existence of u, namely $(1-q)\mathrm{G}^\Omega(h^q V)(x) < h(x)$, $\forall x \in \Omega$. □

Proof of Theorem 3

Suppose $\Omega \subset M$ is a relatively compact domain whose boundary is regular with respect to the Dirichlet problem. Let

$$h = P^{\Omega}\nu + \mathrm{G}^{\Omega}\mu > 0 \quad \text{in } \Omega. \tag{40}$$

Since μ is uniformly bounded in Ω, we have

$$\mathrm{G}^{\Omega}\mu \le (\sup_{\Omega} \mu)\, \mathrm{G}^{\Omega}1,$$

and hence by the regularity of $\partial\Omega$,

$$\lim_{y\to x} \mathrm{G}^{\Omega}\mu(y) = \lim_{y\to x} \mathrm{G}^{\Omega}1(y) = 0, \quad \lim_{y\to x} P^{\Omega}\nu(y) = \nu(x), \quad x \in \partial\Omega.$$

It follows $h \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$, $-\Delta h = \mu$, and

$$\lim_{y\to x} h(y) = \lim_{y\to x} u(y) = \nu(x), \quad x \in \partial\Omega.$$

Proof of Theorem 3(continuation)

For $\epsilon > 0$, set $u_\epsilon = u + \epsilon$, $h_\epsilon = h + \epsilon$, and define the function v_ϵ via

$$\frac{u_\epsilon}{h_\epsilon} = \phi\left(v_\epsilon\right),$$

where ϕ is chosen as in the proof of the previous Lemma. Note that $h_\epsilon > 0$ is superharmonic in Ω, and $-\Delta h_\epsilon = \mu$. Clearly, $h_\epsilon, u_\epsilon, v_\epsilon \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$.
Identity (21) applied to $h_\epsilon, u_\epsilon, v_\epsilon$ in place of h, u, v gives

$$-\Delta(h_\epsilon v_\epsilon) = \frac{-\Delta u}{\phi'(v_\epsilon)} + \frac{\phi''(v_\epsilon)}{\phi'\left(v_\epsilon\right)}|\nabla v|^2 h_\epsilon + \Delta h\left(\frac{\phi(v_\epsilon)}{\phi'\left(v_\epsilon\right)} - v_\epsilon\right),$$

where

$$\phi'(v_\epsilon) = \phi(v_\epsilon)^q = \left(\frac{u_\epsilon}{h_\epsilon}\right)^q.$$

Proof of Theorem 3 (continuation)

Suppose $q > 0$ and $-\Delta u \geq -Vu^q + \mu$, $\mu = -\Delta h$. Hence,

$$-\Delta(h_\epsilon v_\epsilon) \geq -h_\epsilon^q \left(\frac{u}{u_\epsilon}\right)^q V + \frac{\phi''(v_\epsilon)}{\phi'(v_\epsilon)} |\nabla v|^2 h_\epsilon + \Delta h \left(\frac{\phi(v_\epsilon) - 1}{\phi'(v_\epsilon)} - v_\epsilon\right).$$

Drop the last two non-negative terms on the right:

$$-\Delta(h_\epsilon v_\epsilon) + h_\epsilon^q \left(\frac{u}{u_\epsilon}\right)^q V \geq 0.$$

Hence, the function

$$h_\epsilon v_\epsilon + \mathrm{G}^\Omega \left(h_\epsilon^q \left(\frac{u}{u_\epsilon}\right)^q V\right)$$

is superharmonic in Ω, and has non-negative boundary values:

$$h_\epsilon v_\epsilon = (\nu + \epsilon)\phi^{-1}\left(\frac{u+\epsilon}{\nu+\epsilon}\right) \geq (\nu+\epsilon)\phi^{-1}(1) = 0 \quad \text{on } \partial\Omega,$$

since $u \geq \nu$ on $\partial\Omega$, ϕ is increasing, and $\phi(0) = 1$.

Proof of Theorem 3 (continuation)

Consequently, by the maximum principle lemma,

$$h_\epsilon v_\epsilon \geq -\mathrm{G}^\Omega \left(h_\epsilon^q \left(\frac{u}{u_\epsilon} \right)^q V \right) \quad \text{in } \Omega. \tag{41}$$

Since $u \leq u_\epsilon$, this implies

$$h_\epsilon v_\epsilon \geq -\mathrm{G}^\Omega \left(h_\epsilon^q V_+ \right), \tag{42}$$

where, in the case $q > 1$ we additionally have

$$-\frac{\mathrm{G}^\Omega \left(h_\epsilon^q V_+ \right)}{h_\epsilon} \leq -\frac{\mathrm{G}^\Omega \left(h_\epsilon^q \left(\frac{u}{u_\epsilon} \right)^q V \right)}{h_\epsilon} \leq v_\epsilon < \frac{1}{q-1}. \tag{43}$$

Let us show that in the case $q \geq 1$ actually $u > 0$ in Ω. In terms of u_ϵ, estimate (42) gives, for $q \geq 1$,

$$u_\epsilon \geq h_\epsilon(x) \phi \left(-\frac{\mathrm{G}^\Omega \left(h_\epsilon^q V_+ \right)}{h_\epsilon} \right). \tag{44}$$

Proof of Theorem 3 (continuation)

Clearly, $h_\epsilon \downarrow h$, where $h > 0$ by (40). Passing to the limit as $\epsilon \to 0$, we deduce by the dominated convergence theorem, for $q \geq 1$,

$$u \geq h\phi\left(-\frac{\mathrm{G}^{\Omega}\left(h^q V_+\right)}{h}\right) > 0 \quad \text{in } \Omega.$$

Note that here, for $q > 1$, we have a strict inequality

$$-\frac{\mathrm{G}^{\Omega}\left(h^q V_+\right)(x)}{h(x)} < \frac{1}{q-1},$$

since otherwise $u(x) = +\infty$.

Proof of Theorem 3 (continuation)

Hence, in the case $q \geq 1$, we have $u > 0$ in Ω. Consequently $\frac{u}{u_\epsilon} \uparrow 1$ as $\epsilon \downarrow 0$, and by the dominated convergence theorem,

$$\lim_{\epsilon \to 0} \mathrm{G}^{\Omega} \left(h_\epsilon^q \left(\frac{u}{u_\epsilon} \right)^q V \right) = \mathrm{G}^{\Omega} \left(h^q V \right). \tag{45}$$

The main estimate restated in terms of u_ϵ:

$$u_\epsilon \geq h_\epsilon(x) \phi \Big(- \frac{\mathrm{G}^{\Omega} \left(h_\epsilon^q \left(\frac{u}{u_\epsilon} \right)^q V \right)}{h_\epsilon} \Big), \tag{46}$$

where by (43) the right-hand side is well-defined. Passing to the limit as $\epsilon \downarrow 0$, we deduce, for $q \geq 1$,

$$u \geq h \phi \Big(- \frac{\mathrm{G}^{\Omega} \left(h^q V \right)}{h} \Big).$$

For $q > 1$, additionally,

$$- \frac{\mathrm{G}^{\Omega} \left(h^q V \right)}{h} < \frac{1}{q-1}.$$

Proof of Theorem 3 (continuation)

A similar argument applies for $0 < q < 1$, but in this case u can be equal to zero on an open set, so that $\frac{u}{u_\epsilon} \uparrow \chi_{\Omega^+}$ as $\epsilon \downarrow 0$. Passing to the limit in (41) using the dominated convergence theorem as above gives

$$h v \geq -\mathrm{G}^{\Omega}\left(\chi_{\Omega^+} h^q V\right),$$

which is equivalent to the desired lower estimate for u.
In the case $q < 0$, we define the function v_ϵ in a slightly different way, via the equation

$$\frac{u}{h_\epsilon} = \phi(v_\epsilon),$$

where as before $h_\epsilon = h + \epsilon$, so that $-\Delta h_\epsilon = \mu$, and

$$h_\epsilon \, v_\epsilon = \frac{1}{1-q} \, h_\epsilon^q \left(u^{1-q} - h_\epsilon^{1-q}\right) \in C^2(\Omega) \cap C(\overline{\Omega}). \qquad (47)$$

Proof of Theorem 3 (continuation)

Then

$$-\Delta(h_\epsilon v_\epsilon) + h_\epsilon^q \, V \le 0.$$

Since $u \le \nu$ on $\partial\Omega$, it follows

$$h_\epsilon v_\epsilon = \frac{1}{1-q} (\nu+\epsilon)^q \left(u^{1-q} - (\nu+\epsilon)^{1-q}\right) \le 0 \quad \text{on } \partial\Omega.$$

Hence,

$$h_\epsilon v_\epsilon \le -\mathrm{G}^\Omega(h_\epsilon^q V) \quad \text{in } \Omega, \tag{48}$$

or, equivalently,

$$u \le h_\epsilon \left[1 - (1-q)\, \frac{\mathrm{G}^\Omega(h_\epsilon^q V)}{h_\epsilon}\right]^{\frac{1}{1-q}} \quad \text{in } \Omega. \tag{49}$$

Proof of Theorem 3 (continuation)

From the above estimates we deduce

$$-\frac{G^{\Omega}(h_{\epsilon}^{q} V)}{h_{\epsilon}} \geq v_{\epsilon} > -\frac{1}{1-q}, \tag{50}$$

so that the expression in square brackets in is always positive. Moreover,

$$-\frac{G^{\Omega}(h_{\epsilon}^{q} V_{+})}{h_{\epsilon}} + \frac{G^{\Omega}(h^{q} V_{-})}{h} > -\frac{1}{1-q}. \tag{51}$$

Since $q < 0$, we have $h_{\epsilon}^{q} \uparrow h^{q}$ as $\epsilon \downarrow 0$. Using dominated convergence,

$$-\frac{G^{\Omega}(h^{q} V)(x)}{h(x)} \geq -\frac{1}{1-q}. \tag{52}$$

Notice that here $G^{\Omega}(h^{q} V_{+})(x) < +\infty$; otherwise

$$G^{\Omega}(h^{q} V_{\pm})(x) = +\infty,$$

which contradicts the assumption that $G^{\Omega}(h^{q} V)(x)$ is well-defined.

Proof of Theorem 3 (continuation)

Clearly, (49) yields the following inequality at $\boldsymbol{x}$:

$$\boldsymbol{u \leq h_\epsilon \left[1 - (1-q)\,\frac{\mathrm{G}^\Omega(h_\epsilon^q V_+)}{h_\epsilon} + (1-q)\,\frac{\mathrm{G}^\Omega(h^q V_-)}{h}\right]^{\frac{1}{1-q}}}. \quad (53)$$

By the dominated convergence theorem, we obtain the corresponding upper estimate at $\boldsymbol{x}$:

$$\boldsymbol{u(x) \leq h(x) \left[1 - (1-q)\,\frac{\mathrm{G}^\Omega(h^q V)(x)}{h(x)}\right]^{\frac{1}{1-q}}}.$$

Since by assumption $\boldsymbol{u(x) > 0}$, the expression in square brackets must be strictly positive (the desired necessary condition). □

Extensions of Theorem 3

We continue our discussion of pointwise estimates of solutions in the local case for arbitrary domains $\Omega \subseteq M$ (not necessarily relatively compact). Denote by $\partial_\infty M$ the infinity point of the one-point compactification of M. For any open subset $\Omega \subseteq M$ denote by $\partial_\infty \Omega$ the union of $\partial\Omega$ and $\partial_\infty M$, if Ω is not relatively compact (infinite boundary of Ω). We set $\partial_\infty \Omega = \partial\Omega$ if Ω is relatively compact.

Definition

For a function u defined in $\Omega \subseteq M$, we write

$$\lim_{y \to \partial_\infty \Omega} u(y) = 0, \tag{54}$$

if $\lim_{k\to\infty} u(y_k) = 0$ for any sequence $\{y_k\}$ in Ω that converges to a point of $\partial_\infty \Omega$; the latter means, that either $\{y_k\}$ converges to a point on $\partial\Omega$ or diverges to $\partial_\infty M$. In the same way we understand similar equalities and inequalities involving $\limsup$ and $\liminf$.

Local case

For example, if Ω is relatively compact, then (54) means that $\lim_{k\to\infty} u(y_k) = 0$ for any sequence $\{y_k\}$ converging to a point on $\partial\Omega$. If $\Omega = M$ then $\partial\Omega = \emptyset$ and (54) means that $\lim_{k\to\infty} u(y_k) = 0$ for any sequence $y_k \to \partial_\infty M$, that is, for any sequence $\{y_k\}$ that leaves any compact subset of M.

In particular, for $M = \mathbb{R}^n$, (54) is equivalent to $u(y) \to 0$ as $|y| \to \infty$.

We will use the notation

$$\chi_u(x) = \begin{cases} 1, & u(x) > 0, \\ 0, & u(x) \leq 0. \end{cases}$$

Main results: local case

Theorem 4 (Grigor'yan-Verbitsky 2019)

Let (M, m) be an arbitrary weighted manifold. Let $\Omega \subseteq M$ be a connected open subset of M with a finite Green function G^Ω. Suppose $V, f \in C(\Omega)$, where $f \geq 0$, $f \not\equiv 0$ in Ω. Let $u \in C^2(\Omega)$ satisfy

$$\text{in the case } q > 0: \quad -\Delta u + V u^q \geq f \ \text{ in } \Omega, \quad u \geq 0, \tag{55}$$

or

$$\text{in the case } q < 0: \quad -\Delta u + V u^q \leq f \ \text{ in } \Omega, \quad u > 0. \tag{56}$$

Set $h = \mathrm{G}^\Omega f$ and assume that $h < \infty$ in Ω. Assume also that $\mathrm{G}^\Omega(h^q V)(x)$ (respectively $\mathrm{G}^\Omega(\chi_u h^q V)(x)$ in the case $0 < q < 1$) is well-defined for all $x \in \Omega$.

Main results: local case
(continuation)

Theorem 4 (statements (i)-(ii))

Then the following statements hold for all $\boldsymbol{x} \in \boldsymbol{\Omega}$.

(*i*) *If* $\boldsymbol{q = 1}$, *then*

$$u(x) \geq h(x) e^{-\frac{1}{h(x)} \mathrm{G}^{\Omega}(hV)(x)}. \tag{57}$$

(*ii*) *If* $\boldsymbol{q > 1}$, *then necessarily*

$$-(q-1)\, \mathrm{G}^{\Omega}(h^q V)(x) < h(x), \tag{58}$$

and the following estimate holds:

$$u(x) \geq \frac{h(x)}{\left[1 + (q-1)\, \frac{\mathrm{G}^{\Omega}(h^q V)(x)}{h(x)}\right]^{\frac{1}{q-1}}}. \tag{59}$$

Main results: local case

(continuation)

Theorem 4 (statements (iii)-(iv))

(*iii*) *If* $0 < q < 1$, *then*

$$u(x) \geq h(x) \left[1 - (1-q) \frac{G^{\Omega}(\chi_u h^q V)(x)}{h(x)} \right]_+^{\frac{1}{1-q}}. \qquad (60)$$

(*iv*) *If* $q < 0$ *and* $\lim_{y \to \partial_\infty \Omega} u(y) = 0$, *then necessarily (58) holds, and*

$$u(x) \leq h(x) \left[1 - (1-q) \frac{G^{\Omega}(h^q V)(x)}{h(x)} \right]^{\frac{1}{1-q}}. \qquad (61)$$

Remarks. 1. Condition $f \not\equiv 0$ implies $h = G^{\Omega} f > 0$ in Ω.
2. No *boundary conditions* are imposed in the case $q > 0$.

Remarks

(continuation)

Remarks. 3. In the case $q \geq 1$, it follows from (57) and (59) that the condition

$$\mathrm{G}^{\Omega}\,(h^q\,V)\,(x) < +\infty$$

implies $u(x) > 0$. Moreover, if for some $0 < C < \frac{1}{q-1}$ and all $x \in \Omega$,

$$\mathrm{G}^{\Omega}\,(h^q V)\,(x) \leq C\,h\,(x)\,,$$

then $u \geq c\,h$ in Ω with some constant $c = c\,(C, q) > 0$.

4. In the case $0 < q < 1$, the function u can vanish in Ω, but the estimate of u does not depend on the values of V on the set $\{u = 0\}$. This explains the appearance of the factor χ_u and the subscript $+$ on the right-hand side of (60).

5. In the case $q < 0$, the boundary condition $\lim_{y \to \partial_\infty \Omega} u\,(y) = 0$ is essential; without it u does not admit any upper bound.

Main results: local case

(continuation)

The proof of Theorem 4 reduces to Theorem 3 above that deals with relatively compact sets $\Omega \subset M$, using an exhaustion of $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ by means of increasing relatively compact sets Ω_k with smooth boundary, and approximation of f. We omit the details (see [Grigor'yan-Verbitsky 2019], Proof of Theorem 3.1).

In the next theorem we give estimates of solutions u of semi-linear inequalities (55)-(56) with $f \equiv 0$. (Theorem 4 requires that $f \not\equiv 0$.) Such results are applicable to the so-called gauge function for Schrödinger equations ($q = 1$), large solutions for super-linear equations ($q > 1$), or ground state solutions ($-\infty < q < 1$) to the corresponding equations and inequalities in unbounded domains in $\mathbb{R}^n$ or on noncompact manifolds.