Potential Theory and Nonlinear Elliptic Equations
Lecture 3

I. E. Verbitsky

University of Missouri, Columbia, USA

Nankai University, Tianjing, China
June 2021

Publications

1. A. Grigor'yan and I. Verbitsky, *Pointwise estimates of solutions to nonlinear equations for non-local operators*, ***Ann. Scuola Norm. Super. Pisa***, **20** (2020) 721–750
2. A. Grigor'yan and I. Verbitsky, *Pointwise estimates of solutions to semilinear elliptic equations and inequalities*, ***J. D′Analyse Math.***, **137** (2019) 529–558
3. M. Frazier and I. Verbitsky, *Positive solutions and harmonic measure for Schrödinger operators in uniform domains*, ***Pure Appl. Funct. Analysis*** (2021), arXiv:2011.04083
4. A. Grigor'yan and W. Hansen, *Lower estimates for a perturbed Green function*, ***J. D′Analyse Math.***, **104** (2008), 25–58.
5. N. Kalton and I. Verbitsky, *Nonlinear equations and weighted norm inequalities*, ***Trans. Amer. Math. Soc.*** **351** (1999), 3441–3497.
6. H. Brezis and X. Cabré, *Some simple nonlinear PDE's without solutions*, ***Boll. Unione Mat. Ital.***, **8**, Ser. 1-B (1998) 223–262.

Additional literature

1. J. L. Doob, *Classical Potential Theory and Its Probabilistic Counterpart,* Classics in Math., Springer, New York –Berlin–Heidelberg–Tokyo, 2001 (Reprint of the 1984 ed.)
2. A. Grigor'yan, *Heat Kernel and Analysis on Manifolds*, Amer. Math.Soc./Intern. Press Studies in Adv. Math., **47**, 2009.
3. N. S. Landkof, *Foundations of Modern Potential Theory*, Grundlehren der math. Wissenschaften, **180**, Springer, New York–Heidelberg, 1972.

Main results: local case

Recall the following theorem (without boundary data) from Lecture 2.

Theorem 4 (Grigor'yan-Verbitsky 2019)

Let (M, m) be an arbitrary weighted manifold. Let $\Omega \subseteq M$ be a connected open subset of M with a finite Green function G^{Ω}. Suppose $V, f \in C(\Omega)$, where $f \geq 0$, $f \not\equiv 0$ in Ω. Let $u \in C^2(\Omega)$ satisfy

$$\text{in the case } q > 0: \quad -\Delta u + V u^q \geq f \ \text{ in } \Omega, \quad u \geq 0, \tag{1}$$

or

$$\text{in the case } q < 0: \quad -\Delta u + V u^q \leq f \ \text{ in } \Omega, \quad u > 0. \tag{2}$$

Set $h = \mathrm{G}^{\Omega} f$ and assume that $h < \infty$ in Ω. Assume also that $\mathrm{G}^{\Omega}(h^q V)(x)$ (respectively $\mathrm{G}^{\Omega}(\chi_u h^q V)(x)$ in the case $0 < q < 1$) is well-defined for all $x \in \Omega$.

Main results: local case
(continuation)

Theorem 4 (statements (i)-(ii))

Then the following statements hold for all $x \in \Omega$.

(*i*) *If $q = 1$, then*

$$u(x) \geq h(x) e^{-\frac{1}{h(x)} \mathrm{G}^{\Omega}(hV)(x)}. \tag{3}$$

(*ii*) *If $q > 1$, then necessarily*

$$-(q-1)\, \mathrm{G}^{\Omega}(h^q V)(x) < h(x), \tag{4}$$

and the following estimate holds:

$$u(x) \geq \frac{h(x)}{\left[1 + (q-1)\, \frac{\mathrm{G}^{\Omega}(h^q V)(x)}{h(x)}\right]^{\frac{1}{q-1}}}. \tag{5}$$

Main results: local case
(continuation)

Theorem 4 (statements (iii)-(iv))

(*iii*) *If* $\mathbf{0} < q < \mathbf{1}$, *then*

$$u(x) \geq h(x) \left[1 - (1-q) \frac{G^{\Omega}(\chi_u h^q V)(x)}{h(x)} \right]_+^{\frac{1}{1-q}}. \qquad (6)$$

(*iv*) *If* $q < \mathbf{0}$ *and* $\lim_{y \to \partial_\infty \Omega} u(y) = \mathbf{0}$, *then necessarily (4) holds, and*

$$u(x) \leq h(x) \left[1 - (1-q) \frac{G^{\Omega}(h^q V)(x)}{h(x)} \right]^{\frac{1}{1-q}}. \qquad (7)$$

Remarks. 1. Condition $f \not\equiv 0$ implies $h = G^{\Omega} f > 0$ in Ω.
2. No *boundary conditions* are imposed in the case $q > 0$.

Extensions of Theorem 3: local case

The proof of Theorem 4 reduces to Theorem 3 that deals with relatively compact sets $\Omega \subset M$, using an exhaustion of $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ by means of increasing relatively compact sets Ω_k with smooth boundary, and approximation of f. We omit the details (see [Grigor'yan-Verbitsky 2019], Proof of Theorem 3.1).

In the next theorem we give estimates of solutions u of semilinear inequalities with both $\nu \equiv 0$ and $f \equiv 0$. (Theorem 4 requires $f \not\equiv 0$.)

Such results are applicable to the so-called gauge function for Schrödinger equations ($q = 1$), large solutions for super-linear equations ($q > 1$), or ground state solutions ($-\infty < q < 1$) to the corresponding equations and inequalities in unbounded domains in $\mathbb{R}^n$ or on noncompact Riemannian manifolds.

Main results: local case

Theorem 5 (Grigor'yan-Verbitsky 2019)

Let (M, m) *be an arbitrary weighted manifold. Let* $\Omega \subseteq M$ *be a connected open subset of* M *with a finite Green function* G^Ω*. Suppose* $V \in C(\Omega)$*. Let* $u \in C^2(\Omega)$ *satisfy either the inequality*

$$-\Delta u + V\, u^q \ge 0, \quad u \ge 0 \text{ in } \Omega, \qquad \text{if } q > 0, \tag{8}$$

or

$$-\Delta u + V\, u^q \le 0, \quad u > 0 \text{ in } \Omega, \qquad \text{if } q < 0. \tag{9}$$

Assume also that $\mathrm{G}^\Omega V(x)$ *(respectively* $\mathrm{G}^\Omega(\chi_u V)(x)$ *in the case* $0 < q < 1$*) is well-defined for all* $x \in \Omega$*. Then the following statements hold for all* $x \in \Omega$*.*

Main results: local case
(continuation)

Theorem 5 (statements (i)-(ii))

(i) *If* $q = 1$ *and*

$$\liminf_{y \to \partial_\infty \Omega} u(y) \geq 1 \tag{10}$$

then

$$u(x) \geq e^{-\mathrm{G}^{\Omega} V(x)}. \tag{11}$$

(ii) *If* $q > 1$ *and*

$$\lim_{y \to \partial_\infty \Omega} u(y) = +\infty, \tag{12}$$

then necessarily $\mathrm{G}^{\Omega} V(x) > 0$, *and*

$$u(x) \geq \left[(q-1)\, \mathrm{G}^{\Omega} V(x) \right]^{-\frac{1}{q-1}}. \tag{13}$$

Main results: local case
(continuation)

Theorem 5 (statements (iii)-(iv))

(*iii*) *If* $0 < q < 1$, *then*

$$u(x) \geq \left[-(1-q)\, \mathrm{G}^{\Omega}(\chi_u V)(x)\right]_+^{\frac{1}{1-q}}. \tag{14}$$

(*iv*) *If* $q < 0$ *and* $\lim_{y \to \partial_\infty \Omega} u(y) = 0$, *then necessarily* $\mathrm{G}^{\Omega} V(x) \leq 0$, *and*

$$u(x) \leq \left[-(1-q)\, \mathrm{G}^{\Omega} V(x)\right]^{\frac{1}{1-q}}. \tag{15}$$

Remarks. 1. The proof of Theorem 5 is similar to that of Theorem 4, using an exhaustion $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ by increasing relatively compact sets Ω_k, so that $G^{\Omega_k} \uparrow G^{\Omega}$ (see [Grigor'yan-Verbitsky 2019], Proof of Theorem 3.3).

Remarks

(continuation)

2. The **boundary conditions** imposed in the cases $q \geq 1$ and $q < 0$ are essential for the estimates. Stronger two-sided estimates for $q = 1$ [Frazier-Verbitsky 2017/21] if $V \leq 0$, true for $\sigma = -V \in \mathcal{M}^+(\Omega)$.

3. The only case where we impose **no boundary conditions** is in sublinear problems where $0 < q < 1$. If $V \leq 0$, we may assume $\sigma = -V \in \mathcal{M}^+(\Omega)$. Then any nontrivial (generalized) solution $u \geq 0$ to the inequality $-\Delta u \geq \sigma\, u^q$ in Ω is **strictly positive**, and satisfies the estimate

$$u(x) \geq \left[(1-q)\, \mathrm{G}^{\Omega}\sigma(x)\right]^{\frac{1}{1-q}}, \quad x \in \Omega. \tag{16}$$

The constant $(1-q)^{\frac{1}{1-q}}$ in this inequality is sharp.

4. Analogues of (16) for $0 < q < 1$ will be proved below for non-local operators and more general kernels in place of G^{Ω}. Two-sided estimates in the one-dimensional example $\Omega = (0, +\infty)$ discussed in the Introduction.

Nonlinear integral equations with general positive kernel

Non-local case

Let (Ω, m) be a locally compact measure space. The theorems below give some sharp existence results together with pointwise estimates of solutions $0 < u < +\infty$ dm-a.e. (for $q > 1$, $V \leq 0$, or $q < 0$, $V \geq 0$):

$$u(x) + \int_\Omega K(x,y)\, u(y)^q\, V(y)\, dm(y) = h(x) \quad \text{in } \Omega. \tag{17}$$

Here $K : \Omega \times \Omega \to [0, +\infty]$ a Borel measurable ***kernel***. For $\mu \in \mathcal{M}^+(\Omega)$, we set

$$K\mu(x) = \int_\Omega K(x,y)\, d\mu(y).$$

Nonlinear integral equations with general positive kernel
(continuation)

More generally, for $\sigma \in \mathcal{M}^+(\Omega)$ (in place of $d\sigma = -V\,dm$), we consider the equation

$$u = K(u^q\,d\sigma) + h, \quad u \geq 0 \text{ in } \Omega,$$

which serves as an analogue of the equation

$$-\Delta u = \sigma u^q + \mu, \quad u \geq 0 \text{ in } \Omega, \tag{18}$$

where u is a ***generalized*** solution with zero boundary values.

In this case, $K = G^\Omega$ is the Green function of the Laplacian Δ, and $h = \mathrm{G}^\Omega \mu$ is the Green potential of a measure μ in Ω.
For bounded C^2-domains Ω, and $\mu \in L^1(\Omega, \partial_\Omega dx)$ this coincides with the notion of a ***very weak*** solution. Here $\partial_\Omega(x) = \text{dist}(x, \Omega^c)$.

Existence and estimates of solutions ($q > 1$)

Theorem 6 (Kalton-Verbitsky 1999)

Let (Ω, σ) be a locally compact measure space, $K \geq 0$ a kernel, and $h \geq 0$ a measurable function. For $q > 1$, suppose

$$K(h^q d\sigma)(x) \leq \left(1 - \frac{1}{q}\right)^q \frac{1}{q-1} h(x) \quad d\sigma\text{-a.e. in } \Omega. \tag{19}$$

Then $u = K(u^q d\sigma) + h$ has a **minimal** *solution u such that*

$$h(x) \leq u(x) \leq \frac{q}{q-1} h(x) \quad d\sigma\text{-a.e.} \quad \text{in } \Omega. \tag{20}$$

Remarks. 1. The extra constant $\left(1 - \frac{1}{q}\right)^q < 1$ ensures existence and provides an upper bound. 2. A matching ***necessary*** condition holds for Green's kernels (with 1) and quasi-metric kernels. 3. A sharper lower bound holds for **all** solutions u (Theorems 3–5 in the local case).

Existence and estimates of solutions ($q < 0$)

Theorem 7 (Grigor'yan-Verbitsky 2020)

For $q < 0$ and $\sigma, \mu \geq 0$, $h = K\mu$, suppose the following condition holds,

$$K(h^q d\sigma)(x) \leq \left(1 - \frac{1}{q}\right)^q \frac{1}{1-q} h(x) \quad d\sigma\text{-a.e. in } \Omega. \tag{21}$$

Then $u + K(u^q d\sigma) = h$ has a **maximal** *solution u such that*

$$\frac{1}{1 - \frac{1}{q}} h(x) \leq u(x) \leq h(x) \quad d\sigma\text{-a.e.} \quad \text{in } \Omega. \tag{22}$$

Remarks. 1. Theorems 6–7 combined with Theorems 3–5 give ***necessary and sufficient*** conditions for the existence of weak solutions (up to a constant). 2. The constants in (19) and (21) are smaller than the constant $\frac{1}{|q-1|}$ in the ***necessary*** conditions for both $q > 1$ and $q < 0$.

Proof of Theorem 7

Thorem 6 ($q > 1$) is well-known, so we give only a proof of Theorem 7 in the case $q < 0$. Let us assume that

$$K(h^q d\sigma)(x) \le a\, h(x) \quad d\sigma - \text{a.e. in } \Omega,$$

for some constant $a > 0$, where $0 < h < +\infty$ a.e.
Set $u_0 = h$, and construct a sequence of consecutive iterations u_k by

$$u_{k+1} + K(u_k^q d\sigma) = h, \quad k = 0, 1, 2, \ldots.$$

Clearly, by the above inequality,

$$(1 - a)h(x) \le u_1(x) = h(x) - K(h^q d\sigma)(x) \le h(x) = u_0(x).$$

Proof of Theorem 7

(continuation)

We set $b_0 = 1$, $b_1 = 1 - a$, and continue the argument by induction. Suppose that for some $k = 1, 2, \ldots$

$$b_k\, h(x) \le u_k(x) \le u_{k-1}(x) \quad \text{in } \Omega.$$

Since $q < 0$ and $\sigma \ge 0$, we deduce using the above estimates,

$$(1 - a\, b_k^q)\, h(x) \le h(x) - b_k^q\, K(h^q d\sigma)(x) \le h(x) - K(u_k^q d\sigma)(x),$$

where the right-hand side $h - K(u_k^q d\sigma) = u_{k+1}$. Clearly,

$$u_{k+1}(x) \le h(x) - K(u_{k-1}^q d\sigma)(x) = u_k(x).$$

Hence,

$$b_{k+1}\, h(x) \le u_{k+1}(x) \le u_k(x), \quad \text{where } b_{k+1} = 1 - a\, b_k^q.$$

We need to pick $a > 0$ small enough, so that $b_k \downarrow b$, where $b > 0$, and $b = 1 - a\, b^q$.

Proof of Theorem 7

(continuation)

In other words, we are solving the equation

$$\frac{1-x}{a} = x^q$$

by consecutive iterations $b_{k+1} = 1 - a b_k^q$ starting from the initial value $b_0 = 1$. Clearly, this equation has a solution $0 < x < 1$ if and only if $0 < a \le a_*$, where $y = \frac{1-x}{a_*}$ is the tangent line to the convex curve $y = x^q$. Here the optimal value a_* is found by equating the derivatives, and solving the system of equations

$$x_*^q = \frac{1-x_*}{a}, \quad q x_*^{q-1} = -\frac{1}{a_*},$$

which gives

$$a_* = \left(1 - \frac{1}{q}\right)^q \frac{1}{1-q}, \quad x_* = \frac{1}{1-\frac{1}{q}}.$$

Proof of Theorem 7

(continuation)

Letting $a = a_*$, we see that by the convexity of $y = x^q$, there is a unique solution $x_* = \frac{1}{1-\frac{1}{q}}$, and by induction, $x_* < b_{k+1} < b_k < 1$, so that

$$b_k \downarrow b = x_* = \frac{1}{1 - \frac{1}{q}} > 0.$$

From this it follows that the desired inequality holds for all $k = 1, 2, \ldots$. Passing to the limit as $k \to \infty$, and using the monotone convergence theorem shows that $u = \lim_{k\to\infty} u_k$ is a solution of the integral equation such that

$$b\, h(x) \le u(x) \le u_0(x) = h(x).$$

Moreover, it is easy to see by construction that u is a maximal solution, that is, if $\tilde{u}$ is another non-negative solution to (17), then $\tilde{u} \le u_k$ for every $k = 0, 1, 2, \ldots$, and consequently $\tilde{u} \le u$ in Ω. □

Lower estimates for homogeneous equations $(0 < q < 1)$

The weak maximum principle

A kernel K on $\Omega \times \Omega$ satisfies the ***weak maximum principle*** **(WMP)** with constant $\mathfrak{b} \geq 1$ if, for any $\nu \in \mathcal{M}^+(\Omega)$ with compact support,

$$\sup \Big\{ K\nu(y) \colon \; y \in \Omega \Big\} \leq \mathfrak{b} \, \sup \Big\{ K\nu(y) \colon \; y \in \operatorname{supp} \nu \Big\}.$$

We consider the ***homogeneous*** sublinear equation $(0 < q < 1$, $h = 0)$

$$u = K(u^q d\sigma), \quad u > 0 \text{ in } \Omega,$$

where $\sigma \in \mathcal{M}^+(\Omega)$.

This generalizes the sublinear elliptic equation

$$(-\Delta)^{\frac{\alpha}{2}} u = \sigma \, u^q \quad \text{in } \mathbb{R}^n, \quad \liminf_{x \to \infty} u = 0,$$

for $0 < \alpha < n$, or in $\Omega \subset \mathbb{R}^n$ with $0 < \alpha \leq 2$, $u = 0$ in Ω^c.

Lower estimates for homogeneous equations $(0 < q < 1)$ (continuation)

Theorem 8 (Grigor'yan-Verbitsky 2020)

Let $0 < q < 1$, (Ω, σ) a locally compact measure space. Let K be a non-negative kernel on $\Omega \times \Omega$ which satisfies the **(WMP)**. *Then any nontrivial nonnegative solution u to $u \ge K(u^q d\sigma)$ satisfies*

$$u(x) \ge (1-q)^{\frac{1}{1-q}} \mathfrak{b}^{-\frac{q}{1-q}} \Big[K\sigma(x) \Big]^{\frac{1}{1-q}} \quad d\sigma\text{-a.e. in } \Omega. \tag{23}$$

Remarks. 1. The constant $(1-q)^{\frac{1}{1-q}}$ in the case $\mathfrak{b} = 1$ is sharp.

2. Lower estimate in Theorem 8 fails without the **(WMP)**.

3. Lower estimate holds for ***all*** $x \in \Omega$: $K(u^q d\sigma)(x) \le u(x) < +\infty$.

4. There are analogues for inhomogeneous equations, $\forall\, q \in \mathbb{R} \setminus \{0\}$.

Non-local case, inhomogeneous equations

Let K be a kernel on $\Omega \times \Omega$. Consider the inhomogeneous integral equation

$$u = K(u^q d\sigma) + h, \quad u > 0 \text{ in } \Omega,$$

where $\sigma \in \mathcal{M}^+(\Omega)$, and $h \geq 0$ ($h \not\equiv 0$).

This is a generalization of the semilinear elliptic equation

$$(-\Delta)^{\frac{\alpha}{2}} u = \sigma\, u^q + \mu \quad \text{in } \mathbb{R}^n, \quad \liminf_{x \to \infty} u = 0,$$

for $0 < \alpha < n$, or in Ω, $0 < \alpha \leq 2$, $u = 0$ in Ω^c, $h = G^\alpha \mu$, $\mu \geq 0$.
We introduce the ***modified kernel***

$$\widetilde{K}(x,y) = \frac{K(x,y)}{h(x)\, h(y)}, \quad x, y \in \Omega.$$

The weak domination principle

Let $h : \Omega \to (0, +\infty]$ be a lower semicontinuous function on Ω. Let $K : \Omega \times \Omega \to [0, +\infty]$ be a lower semicontinuous kernel. Then K satisfies the ***weak domination principle*** **(WDP)** with respect to h if:
For any compactly supported $\nu \in \mathcal{M}^+(\Omega)$ *and any constant* $M > 0$,

$$K\nu(x) \le M\, h(x), \, \forall\, x \in \operatorname{supp}(\nu) \implies K\nu(x) \le \mathfrak{b}\, M\, h(x), \, \forall\, x \in \Omega,$$

whenever $K\nu$ is bounded (or ν has finite energy: $\int_\Omega K\nu d\nu < +\infty$).

Remark. The kernel K satisfies the **(WDP)** if the modified kernel $\widetilde{K}$ satisfies the **(WMP)** provided for any compactly supported $\nu \in \mathcal{M}^+(\Omega)$ there exist compactly supported $\nu_n \in \mathcal{M}^+(\Omega)$, $K\nu_n \in C(\Omega)$, $K\nu_n \uparrow K\nu$ in Ω.

Non-local case, main theorem

Theorem 9 (Grigor'yan-Verbitsky 2020)

Let $h > 0$ *be a lower semicontinuous function in* Ω. *Let* K *be a kernel in* $\Omega \times \Omega$ *such that the* $(\mathbf{WMP})$ *holds for* $\widetilde{K}$, h. *Suppose that* $u \geq 0$ *satisfies* $u \geq K(u^q d\sigma) + h$ *if* $q > 0$, *and the opposite if* $q < 0$.
(i) If $q > 0$ $(q \neq 1)$, *we have*

$$u(x) \geq h(x) \left\{ 1 + \mathfrak{b} \left[\left(1 + \frac{(1-q)\, K(h^q d\sigma)(x)}{\mathfrak{b}\, h(x)} \right)^{\frac{1}{1-q}} - 1 \right] \right\}, \quad (24)$$

where in the case $q > 1$ *necessarily*

$$K(h^q d\sigma)(x) < \frac{\mathfrak{b}}{q-1}\, h(x), \quad (25)$$

for all $x \in \Omega$ *such that* $K(u^q d\sigma)(x) + h(x) \leq u(x) < +\infty$.

Non-local case, main theorem

(continuation)

Theorem 9 (statements (ii), (iii))

(ii) In the case $q = 1$,

$$u(x) \geq h(x)\left[1 + \mathfrak{b}\left(e^{\mathfrak{b}^{-1}\,\frac{K(hd\sigma)(x)}{h(x)}} - 1\right)\right], \quad x \in \Omega. \tag{26}$$

(iii) If $q < 0$, *then*

$$u(x) \leq h(x)\left\{1 - \mathfrak{b}\left[1 - \left(1 - \frac{(1-q)\,K(h^q d\sigma)(x)}{\mathfrak{b}\,h(x)}\right)^{\frac{1}{1-q}}\right]\right\}, \tag{27}$$

for $x \in \Omega$, *and necessarily*

$$K(h^q d\sigma)(x) < \frac{\mathfrak{b}}{1-q}\left[1 - (1 - \mathfrak{b}^{-1})^{1-q}\right] h(x), \tag{28}$$

for all $x \in \Omega$: $0 < u(x) + K(u^q d\sigma)(x) \leq h(x) < +\infty$.

Some additional references

1. **Linear case $q = 1$** (Schrödinger equations): lower estimates of perturbed Green's functions on domains and manifolds for $\sigma = -V \leq 0$ [Grigor'yan-Hansen 2008]. For $\sigma \geq 0$, [Frazier-Verbitsky 2017], [Frazier-Nazarov-Verbitsky 2014] two-sided estimates of perturbed Green's functions, ***quasimetric kernels*** K, arbitrary $\sigma \geq 0$ (under the spectrum of the Schrödinger operator). [Murata 1986], [Pinchover 2007] nice σ.
2. **Superlinear case $q > 1$**: For $\sigma \geq 0$, [Brezis-Cabré 1998] (for the Laplacian $-\Delta$ only), [Kalton-Verbitsky 1999] two-sided estimates (quasimetric kernels, but no sharp constants).
3. **Sublinear case $0 < q < 1$**: $\sigma \geq 0$, ***bounded*** solutions, $-\Delta$ on $\mathbb{R}^n$ [Brezis-Kamin 1992]; two-sided estimates [Cao-Verbitsky 2017]; existence of weak solutions, **(WMP)**-kernels [Quinn-Verbitsky 2018] .
4. **Negative exponents:** $q < 0$, only $\sigma = \pm\, \partial_\Omega(x)^{-\beta}$ ($\beta > 0$) $\partial_\Omega(x) = \mathrm{dist}(x, \partial\Omega)$ [Dupaigne-Ghergu-Radulescu 2007].

Nonlinear integral inequalities

The proofs of Theorem 8 and Theorem 9 are given below.
Let Ω be locally compact (possibly totally discrete), $\sigma \in \mathcal{M}^+(\Omega)$, $K \geq 0$ a kernel on $\Omega \times \Omega$. Consider the nonlinear inequality

$$u(x) \geq \mathcal{A}u(x) + 1 \quad d\sigma - \text{a.e. in } \Omega,$$

where $\mathcal{A}$ is the nonlinear map

$$\mathcal{A}u = K\Big(g(u)d\sigma\Big), \quad 1 \leq u < +\infty \ d\sigma - a.e.$$

Here $g\colon [1, a) \to (0, +\infty)$, is non-decreasing, continuous, where $a \in (1, +\infty]$. Let $g(a) = \lim_{t \to a^-} g(t) \in (0, +\infty]$, and extend g from $[1, a]$ to $[1, +\infty]$, by setting $g(t) := g(a)$ for $a \leq t \leq +\infty$.

Our goal: sharp lower estimates of u, better than the trivial $u \geq 1$.

We assume $\alpha := g(1) > 0$. In the case $\alpha = 0$, a simple example: $g(t) = \log t \ (t \geq 1)$, $u \equiv 1$ shows no self-improving estimates.

Nonlinear integral inequalities

(continuation)

Remark. Since $\alpha = g(1) > 0$, WLOG we assume $\alpha = 1$, so that

$$g\colon [1, \infty] \to [1, +\infty], \quad g(1) = 1.$$

It is convenient to define a new measure:

$$d\nu = g(u)\, d\sigma, \quad \text{so that} \quad K\nu = \mathcal{A}u,$$

and a new function $\phi\colon [0, +\infty] \to [1, +\infty]$ continuous non-decreasing,

$$\phi(t) = g(t+1), \quad \phi(0) = 1.$$

Observe that since $u \geq \mathcal{A}u + 1$, we have

$$d\nu = g(u) d\sigma \geq g(\mathcal{A}u + 1) d\sigma = \phi(K\nu)\, d\sigma.$$

Iterating the preceding inequality, we obtain

$$d\nu \geq \phi(K\nu)\, d\sigma \geq \phi\Big(K\Big(\phi(K\nu)\, d\sigma\Big) d\sigma\Big) \geq \ldots.$$

Nonlinear integral inequalities
(continuation)

Notice that $K\nu \geq K\sigma$, since

$$\phi(0) = g(1) \geq 1.$$

Then $\phi(K\nu) \geq \phi(K\sigma)$, and consequently,

$$u \geq 1 + K\nu \geq 1 + K\Big(\phi(K\nu) d\sigma\Big) \geq \ldots \geq 1 + K\sigma_j,$$

where $j = 1, 2, \ldots$, and σ_j is defined by induction: $\sigma_0 = \sigma$, and

$$d\sigma_j = \phi(K\sigma_{j-1})\, d\sigma, \quad j \geq 1.$$

We next prove a series of lemmas in order to estimate

$$K\sigma_j = K\left[\phi(K\sigma_{j-1})\, d\sigma\right], \quad j = 1, 2, \ldots.$$

A key real variable (rearrangements) lemma

Lemma (rearrangements)

Let (Ω, σ) *be a* σ*-finite measure space, and let* $a = \sigma(\Omega) \leq +\infty$. *Let* $f : \Omega \to [0, +\infty]$ *be a measurable function. Let* $\phi\colon [0, a) \to [0, +\infty)$ *be a continuous, monotone non-decreasing function, and set* $\phi(a) := \lim_{t \to a^-} \phi(t) \in (0, +\infty]$. *Then the following inequality holds:*

$$\int_0^{\sigma(\Omega)} \phi(t)\, dt \leq \int_\Omega \phi\left(\sigma\left(\{z \in \Omega\colon\ f(z) \leq f(y)\}\right)\right)\, d\sigma(y).$$

Proof: Reduction to discrete case, rearrangement in non-decreasing order.

A key potential theory (integration by-parts) lemma

If $\phi\colon [0, a) \to [0, +\infty)$ is non-decreasing continuous, we can extend it to $[0, +\infty]$ by $\phi(t) := \lim_{s\to a^-} \phi(s)$ for $t \in [a, +\infty]$. Here $a \in [0, +\infty]$. So WLOG we may assume ϕ is defined on $[0, +\infty]$.

Lemma (by-parts)

Suppose $\nu \in \mathcal{M}^+(\Omega)$, $x \in \Omega$. Let $a := \nu(\Omega) \in (0, +\infty]$. Suppose K is a non-negative **(WMP)***-kernel with $\mathfrak{b} \geq 1$, and $\phi\colon [0, +\infty] \to [0, +\infty]$ is non-decreasing, continuous. Then*

$$\int_0^{K\nu(x)} \phi(t)dt \leq K\Big[\phi(\mathfrak{b}\, K\nu)d\nu\Big](x).$$

Idea of the proof: Fix $x \in \Omega$. Use the rearrangements lemma with $d\nu = K(x, \cdot)d\sigma$, $f(y) = K\nu(y)$, and apply the **(WMP)** appropriately. The details are given below.

Proof of the by-parts lemma

Fix $x \in \Omega$, and suppose first $K\nu(x) < \infty$. WLOG assume that $K\nu(x) > 0$. For any $y \in \Omega$, set

$$E_y = \{z \in \Omega \colon \; K\nu(z) \le \; K\nu(y)\}.$$

Clearly,

$$K\nu_{E_y}(w) \le K\nu(w) \le K\nu(y) \quad \text{for all } w \in E_y.$$

Hence by the **(WMP)** applied to ν_{E_y} (WLOG assume E_y is compact),

$$K\nu_{E_y}(w) \le \mathfrak{b}\, K\nu(y) \quad \text{for all } w \in \Omega.$$

In particular, with $w = x$,

$$K\nu_{E_y}(x) = \int_{E_y} K(x, z)\, d\nu(z) \le \mathfrak{b}\, K\nu(y).$$

Proof of the by-parts lemma
(continuation)

Let $f(y) = K\nu(y)$, then $E_y = \{z \in \Omega : \ f(z) \leq f(y)\}$.
Now let $d\sigma(y) = K(x, y)\, d\nu(y)$, so that $\sigma(\Omega) = K\nu(x)$.
Then by the rearrangements lemma and the preceding estimate,

$$\begin{aligned}
\int_0^{K\nu(x)} \phi(t)\, dt &\leq \int_\Omega \phi\Big(\int_{E_y} d\sigma(z)\Big) d\sigma(y) \\
&= \int_\Omega \phi\Big(\int_{E_y} K(x, z)\, d\nu(z)\Big) K(x, y)\, d\nu(y) \\
&\leq K\Big[\phi\Big(\mathfrak{b}\, K\nu\Big) d\nu\Big](x).
\end{aligned}$$

Proof of the by-parts lemma

(continuation)

In the remaining case $K\nu(x) = +\infty$, let us show that $K\left[\phi(\mathfrak{b}\, K\nu)d\nu\right](x) = +\infty$ as well. Denote by E the set of all points $y \in \Omega$ for which $K\nu(y) \leq 1$ (assume WLOG E is compact). Then

$$K\nu_E(y) \leq 1, \quad \text{for all } y \in E.$$

Hence, by the **(WMP)** applied to ν_E,

$$K\nu_E(w) \leq \mathfrak{b} \quad \text{for all } w \in \Omega.$$

In particular, $K\nu_E(x) \leq \mathfrak{b}$, and so

$$K\nu_{E^c}(x) = +\infty.$$

Notice that $K\nu(y) > 1$ for all $y \in E^c$. Thus,

$$\begin{aligned} K\left[\phi(\mathfrak{b}K\nu)d\nu\right](x) &\geq K\left[\phi(\mathfrak{b}K\nu)d\nu_{E^c}\right](x) \\ &\geq \phi(\mathfrak{b})\, K\nu_{E^c}(x) = +\infty. \quad \square \end{aligned}$$

Iterated by-parts lemma

Suppose $\phi\colon [0,+\infty) \to [0,+\infty]$ is a non-decreasing continuous function. For $\nu \in \mathcal{M}^+(\Omega)$, let $f_1 := K\nu$, $d\nu_1 := \phi(f_1)\, d\nu$, and

$$f_k := K\left(\phi(f_{k-1})d\nu\right), \quad k = 2, 3, \ldots, \tag{29}$$

$$d\nu_k := \phi(f_k)\, d\nu = \phi(K\nu_{k-1})d\nu, \quad k = 2, 3, \ldots. \tag{30}$$

Consequently, $f_1 = K\nu$, $f_2 = K\nu_1 = K\left(\phi(K\nu)d\nu\right)$, and

$$f_k = K\nu_{k-1} = K\left(\phi[K(\cdots[\phi(K\nu)d\nu]\cdots)d\nu]d\nu\right).$$

Iterated by-parts lemma

Lemma (iterations)

Let $\nu \in \mathcal{M}^+(\Omega)$, K, ϕ *satisfy the assumptions of the preceding Lemma. Set*

$$\psi(t) := \phi(\mathfrak{b}^{-1} t), \quad t \geq 0.$$

Then for all $x \in \Omega$,

$$\psi_j\left(K\nu(x)\right) \leq K\nu_j(x), \quad j = 1, 2, \ldots,$$

where $d\nu_j = \phi(K\nu_{j-1})d\nu$ *are defined by iterations, and*

$$\psi_j(t) := \int_0^t \psi \circ \psi_{j-1}(s) ds, \quad \psi_0(t) := t, \quad t \geq 0.$$

Proof: Repeated use of the **(WMP)** and the by-parts lemma. See details in [Grigor'yan-Verbitsky 2020], Lemma 2.7.

Corollary: $\phi(t) = t^q$, $q > 0$

The following is immediate from the iterations Lemma.

Corollary (special case)

Suppose $\nu \in \mathcal{M}^+(\Omega)$, and K is a **(WMP)***-kernel with $\mathfrak{b} \geq 1$. If $q > 0$, then, for all $x \in \Omega$ and $j \geq 1$,*

$$\Big[K\nu(x)\Big]^{1+q+\cdots+q^j} \leq c(q,j)\, \mathfrak{b}^{q(1+q+\cdots+q^{j-1})} K\nu_j(x),$$

where

$$c(q,j) = \prod_{k=1}^{j} (1 + q + \cdots + q^k)^{q^{j-k}}.$$

In particular, in the case $q = 1$, for all $x \in \Omega$ and $j \geq 1$ we have

$$\Big[K\nu(x)\Big]^{j+1} \leq (j+1)!\, \mathfrak{b}^j \, K\nu_j(x).$$

Remark. A direct proof by induction gives constants that grow too fast.

Proof of Theorem 8 ($0 < q < 1$)

Let $u \geq K(u^q d\sigma)$ $d\sigma$-a.e. For $a > 0$, set

$$E_a = \{y \in \Omega\colon \ u(y) \geq a\}.$$

Let $d\nu = \chi_{E_a} d\sigma$. Suppose $u(x) \geq K(u^q d\sigma)(x)$, where $x \in \Omega$. Then

$$u(x) \geq K(u^q d\sigma)(x) \geq a^q K\nu(x), \quad x \in \Omega.$$

Iterating this inequality, as in the iterated potential theory lemma, we obtain

$$u(x) \geq a^{q^{k+1}} K\nu_k(x),$$

where ν_k is defined by (30) with $\phi(t) = t^q$, i.e., $d\nu_1 = (K\nu)^q d\nu$,

$$d\nu_k := (K\nu_{k-1})^q d\nu, \quad k = 2, 3, \ldots.$$

Hence, by the Corollary,

$$u(x) \geq c(q,k)^{-1}\, a^{q^{k+1}} \mathfrak{b}^{-q(1+q+\cdots+q^{k-1})} \, (K\nu(x))^{1+q+\cdots+q^k}.$$

Proof of Theorem 8

(continuation)

Notice that, since $0 < q < 1$,

$$c(q,k) = \prod_{j=1}^{k} (1 + q + \cdots + q^j)^{q^{k-j}}$$
$$< \prod_{j=1}^{k} (1-q)^{-q^{k-j}} < (1-q)^{-(1-q)^{-1}}.$$

Consequently,

$$u(x) \geq (1-q)^{(1-q)^{-1}} a^{q^{k+1}} \mathfrak{b}^{-q(1+q+\cdots+q^{k-1})} (K\nu(x))^{1+q+\cdots+q^k}.$$

Letting $k \to +\infty$, we obtain

$$u(x) \geq (1-q)^{(1-q)^{-1}} \mathfrak{b}^{-\frac{q}{1-q}} (K\nu(x))^{\frac{1}{1-q}}.$$

Finally, letting $a \to 0^+$ yields (23) by the monotone convergence theorem.

Integral inequalities for nondecreasing nonlinearities

Let $g : [1, a) \to [1, +\infty)$ be a nondecreasing, continuous function.
We set

$$F(t) = \int_1^t \frac{ds}{g(s)}, \quad t \geq 1. \tag{31}$$

Here F is defined on $[1, \infty)$. The inverse function F^{-1} is defined on $[0, a)$, and takes values in $[1, \infty)$, where

$$a := \int_1^{+\infty} \frac{ds}{g(s)}. \tag{32}$$

The following theorem is deduced from the iterations lemma and some ODE techniques.

Integral inequalities for nondecreasing nonlinearities

Theorem 10 (lower estimate)

Let $\sigma \in \mathcal{M}^+(\Omega)$, and let K be a **(WMP)***-kernel on Ω with constant $\mathfrak{b} \geq 1$. Let $g\colon [1, +\infty) \to [1, +\infty)$ be nondecreasing, continuous. If $\mathcal{A}u = K(g(u)d\sigma)$, and $u \geq \mathcal{A}u + 1$ $d\sigma$-a.e., then*

$$u(x) \geq 1 + \mathfrak{b} \left[F^{-1} \left(\mathfrak{b}^{-1} K\sigma(x) \right) - 1 \right], \qquad (33)$$

for all $x \in \Omega$ such that $\mathcal{A}u(x) + 1 \leq u(x) < +\infty$, where necessarily

$$\mathfrak{b}^{-1} K\sigma(x) < \ a := \int_1^{+\infty} \frac{ds}{g(s)}. \qquad (34)$$

Remark. We will give below a proof of Theorem 10. A similar proof of Theorem 11 for noninreasing g will be omitted.

Special cases

We now consider some special cases of Theorem 10 for $g(t) = t^q$.

Corollary

Let $q > 0$. Under the assumptions of Theorem 10, suppose u satisfies

$$u \geq K(u^q d\sigma) + 1 \quad d\sigma\text{-}a.e.$$

If $q \neq 1$, then the following inequality holds:

$$u(x) \geq 1 + \mathfrak{b}\Big[\Big(1 + (1-q)\mathfrak{b}^{-1} K\sigma(x)\Big)^{\frac{1}{1-q}} - 1\Big],$$

$$\textit{where necessarily} \quad K\sigma(x) < \frac{\mathfrak{b}}{q-1} \quad \textit{if} \quad q > 1,$$

for all $x \in \Omega$ such that $K(u^q d\sigma)(x) + 1 \leq u(x) < +\infty$. If $q = 1$, then

$$u(x) \geq 1 + \mathfrak{b}\left(e^{\mathfrak{b}^{-1} K\sigma(x)} - 1\right).$$

Integral inequalities for nonincreasing nonlinearities

Theorem 11 (upper estimate)

Let $\sigma \in \mathcal{M}^+(\Omega)$, *and let* K *be a* $\mathbf{(WMP)}$*-kernel (with constant* $\mathfrak{b} \geq 1$*).*
Let $g\colon (0,1] \to [1, +\infty)$ *be nonincreasing, continuous on* $(0,1]$*. Set*

$$F(t) = \int_t^1 \frac{ds}{g(s)}, \quad 0 \leq t \leq 1.$$

If $\mathcal{A}u = K(g(u)d\sigma)$, *and* $0 \leq u \leq -\mathcal{A}u + 1$ $d\sigma$*-a.e., then*

$$u(x) \leq 1 - \mathfrak{b}\left[1 - F^{-1}\Big(\mathfrak{b}^{-1} K\sigma(x)\Big)\right],$$

and the following necessary condition holds:

$$K\sigma(x) < \mathfrak{b}\, F(1 - \mathfrak{b}^{-1}) = \mathfrak{b} \int_{1-\mathfrak{b}^{-1}}^1 \frac{ds}{g(s)},$$

for all $x \in \Omega$ *such that* $0 < u(x) \leq -\mathcal{A}u(x) + 1$.

Integral inequalities in special cases

We now consider the special case $g(t) = t^q$, $q < 0$.

Corollary

Let $q < 0$. *Under the assumptions of Theorem 11, suppose* u *satisfies*

$$0 \le u \le -K(u^q d\sigma) + 1 \quad d\sigma\text{-}a.e.$$

Then the following inequality holds:

$$0 < u(x) \le -\mathfrak{b}\left[\left(1 + (1-q)\mathfrak{b}^{-1} K\sigma(x)\right)^{\frac{1}{1-q}} - 1\right] + 1,$$

$$\text{where necessarily} \quad K\sigma(x) < \frac{\mathfrak{b}}{1-q}\left[1 - (1 - \mathfrak{b}^{-1})^{1-q}\right],$$

for all $x \in \Omega$ *such that* $0 < u(x) \le -K(u^q d\sigma)(x) + 1$.